# On the Consistency/Renormalizability of Relativistic Quantum Field Theories

**1) Metric, $\varepsilon$-tensor, Lorentz-transformation, Pauli– and $\gamma$–matrices:**

Let $g_{\mu\nu} = g^{\mu\nu}$ $(\mu, \nu \in \{0,1,2,3\})$ be the metric tensor and its inverse and

$$\delta^{\,\mu_1\ldots\mu_n}_{\,\nu_1\ldots\nu_n} := \begin{vmatrix} g^{\,\mu_1}_{\,\nu_1} & \cdots & g^{\,\mu_1}_{\,\nu_n} \\ \vdots & & \vdots \\ g^{\,\mu_n}_{\,\nu_1} & \cdots & g^{\,\mu_n}_{\,\nu_n} \end{vmatrix} \quad , \quad \varepsilon^{\,\mu\nu\rho\sigma} := \delta^{\,\mu\nu\rho\sigma}_{\,0123} \; .$$

Remember that a Lorentz-transformation $\Lambda^\mu{}_\nu$ is defined by $\Lambda^\mu{}_\rho \, g_{\mu\nu} \, \Lambda^\nu{}_\sigma = g_{\rho\sigma}$. Let $1_2$ and $1_4$ be the 2– and 4–dimensional unit matrices. Further define the $2\times 2$-matrices $\sigma^1$, $\sigma^2$, $\sigma^2$ ("Pauli–matrices") and the $4\times 4$-matrices $\alpha^1$, $\alpha^2$, $\alpha^3$, $\beta = \gamma^0$, $\gamma^5 = \gamma_5$ by:

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\vec{\alpha} := \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta := \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad \gamma^5 := \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}$$

and the $4\times 4$ matrices $\vec{\gamma} := \gamma^0\,\vec{\alpha}$ and $\sigma^{\,\mu\nu} := (i/2)[\gamma^{\,\mu}, \gamma^{\,\nu}]_-$ with $[A,B]_\pm := AB \pm BA$.

(a) Determine $\varepsilon^{\,0123}$, $\varepsilon_{\,0123}$ and $\delta^{\,\mu_1\ldots\mu_n}_{\,\nu_1\ldots\nu_n}\,\delta^{\,\nu_1\ldots\nu_n}_{\,\rho_1\ldots\rho_n}$, $\varepsilon^{\,\mu\nu\rho\sigma}\varepsilon_{\,\mu\nu\rho\sigma}$, $\varepsilon^{\,\mu\nu\rho\sigma'}\varepsilon_{\,\mu\nu\rho\sigma}$, $\varepsilon^{\,\mu\nu\rho'\sigma'}\varepsilon_{\,\mu\nu\rho\sigma}$, $\varepsilon^{\,\mu\nu'\rho'\sigma'}\varepsilon_{\,\mu\nu\rho\sigma}$, $\varepsilon^{\,\mu'\nu'\rho'\sigma'}\varepsilon_{\,\mu\nu\rho\sigma}$.

(b) How does $\varepsilon^{\,\mu\nu\rho\sigma}$ transform under Lorentz-transformations?

(c) Determine from $[\gamma^\mu, \gamma^\nu]_+ = 2\,g^{\mu\nu}$ the metric underlying the given system of $\gamma$-matrices. Show that $[\gamma^\mu, \gamma^5]_+ = 0$, $(\gamma_5)^2 = 1_4$, $\gamma_5 = -\frac{i}{4!}\,\varepsilon_{\,\mu\nu\rho\sigma}\gamma^{\,\mu}\gamma^{\,\nu}\gamma^{\,\rho}\gamma^{\,\sigma}$, $\gamma_5\,\sigma^{\,\mu\nu} = \frac{i}{2}\,\varepsilon^{\,\mu\nu\rho\sigma}\,\sigma_{\,\rho\sigma}$.

(d) Determine $\mathrm{tr}[\gamma^{\,\mu_1}\ldots\gamma^{\,\mu_n}]$, $\mathrm{tr}[\gamma^5\gamma^{\,\mu_1}\ldots\gamma^{\,\mu_n}]$; show $\mathrm{tr}[\gamma^{\,\mu_1}\ldots\gamma^{\,\mu_n}] = \mathrm{tr}[\gamma^{\,\mu_n}\ldots\gamma^{\,\mu_1}]$.

(e) Show for $C := i\,\gamma^2\gamma^0$ that $C = -\,C^{-1} = -C^{+} = -\,C^{T} = C^{*}$, $(\gamma^{\,\mu})^T = C\gamma^{\,\mu}C$.

**2) Poincaré-algebra:**

The generators of translations are $\partial^\mu = \frac{\partial}{\partial x_\mu}$. The generators of Lorentz-transformations are $\partial^{\,\mu\nu} := x^{\,\mu}\partial^{\,\nu} - x^{\,\nu}\partial^{\,\mu}$. Together they form a closed algebra, i.e. the Poincaré-algebra.

(a) Verify $[AB, C]_- = A[B,C]_- + [A,C]_- B$ and $[A, BC]_- = [A,B]_- C + B[A,C]_-$.

(b) Determine $[\partial^{\,\mu}, \partial^{\,\nu}]_-$, $[\partial^{\,\mu\nu}, \partial^{\,\sigma}]_-$, $[\partial^{\,\mu\nu}, \partial^{\,\sigma\rho}]_-$ in terms of $\partial^{\,\mu}$, $\partial^{\,\mu\nu}$ and the metric.

**3) Lorentz-boost for arbitrary complex 4-vectors $\xi^\mu$ with $\xi^2 = \xi^\mu\xi_\mu \neq 0$:**

For the metric of problem 1) we can define the following Lorentz-boost matrix:

$$\Lambda^{\,\mu}{}_{\nu}(\xi) := \begin{pmatrix} \frac{\xi^0}{\sqrt{\xi^2}} & \frac{\vec{\xi}^{\,T}}{\sqrt{\xi^2}} \\ & \\ \frac{\vec{\xi}}{\sqrt{\xi^2}} & 1_3 + \frac{\vec{\xi}\,\vec{\xi}^{\,T}}{\sqrt{\xi^2}(\sqrt{\xi^2}+\xi^0)} \end{pmatrix} .$$

(a) Show that $\Lambda^{\,\mu}{}_{\nu}(\xi)$ is a Lorentz-transformation.

(b) Show that $\xi^{\,\mu} = \Lambda^{\,\mu}{}_{\nu}(\xi)\;(\sqrt{\xi^2}, \vec{0}\,)^\nu$ and $(\sqrt{\xi^2}, \vec{0}\,)_\nu = \xi_{\,\mu}\Lambda^{\,\mu}{}_{\nu}(\xi)$.

(c) Determine the determinant of $\Lambda^{\,\mu}{}_{\nu}(\xi)$.

(d) For a particle of complex mass $M = m - i\,\frac{\Gamma}{2}$ the mass-shell condition is $p^2 = M^2$. What 4-momentum-hypersurfaces are described by this condition? What's the trajectory in complex space-time of this particle, if it starts at (complex) $x^{\,\mu}$ and propagates with real momentum $\vec{p}$ along the real time axis ($p^\mu x_\mu =$Lorentz-scalar)?

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**4) The Baker-Campbell-Hausdorff relation (=BCHR):**
Prove the following identities for the operators (or matrices) $A$ and $B$:

(a) $e^A\, B\, e^{-A} = B + \frac{1}{1!}[A,B]_- + \frac{1}{2!}[A,[A,B]_-]_- + \frac{1}{3!}[A,[A,[A,B]_-]_-]_- + \ldots$ (= BCHR).

(b) $e^{A+B} = e^A\, e^B\, e^{-\frac{1}{2}[A,B]_-}$ for $[A,[A,B]_-]_- = [B,[A,B]_-]_- = 0$.

(c) $\frac{\partial A^{-1}}{\partial \lambda} = -\,A^{-1}\frac{\partial A}{\partial \lambda}A^{-1}$, if $A$ invertible ($\lambda$ = arbitrary parameter).

For original literature on this issue see: J.E. Campbell, Proc. Lond. Math. Soc. **29** (1898) 14; H.F. Baker, Proc. Lond. Math. Soc. Ser. **3** (1904) 24; F. Hausdorff, Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math.-Naturwiss. Klasse **58** (1906) 19.

**5) Lagrangian describing (complex) two and one frequency systems:**
Let $\Omega_1$ and $\Omega_2$ be two constant complex frequencies and $z(t)$ and $\dot{z}(t)$ one complex degree of freedom and its velocity, whose time-dependence is determined by the following Lagrangian (see H. Bateman, Phys. Rev. 38 (1931) 815 and G. Vitiello, hep-th/0110182):

$$L(z,\dot{z}) = \frac{1}{2}\, e^{-\,i\,(\Omega_1+\Omega_2)\,t}\left[\dot{z}^2 - i\,(\Omega_1+\Omega_2)\,\dot{z}\,z - \frac{1}{2}\,(\Omega_1^2+\Omega_2^2)\,z^2\right] .$$

(a) Show, that the Lagrange equations of motion of $L(z,\dot{z})$ are:

$$\left[\frac{\partial}{\partial t} - i\,\Omega_1\right]\left[\frac{\partial}{\partial t} - i\,\Omega_2\right] z(t) = 0 \quad \Leftrightarrow \quad \left[\frac{\partial^2}{\partial t^2} - i\,(\Omega_1+\Omega_2)\frac{\partial}{\partial t} - \Omega_1\,\Omega_2\right] z(t) = 0\ .$$

(b) What is the general solution of the Lagrange equation of motion? How to obtain the corresponding Green-function of the differential operator?

(c) What is the Hamiltonian $H(z,p)$ corresponding to $L(z,\dot{z})$? What are the corresponding Hamilton equations of motion? Is it possible to define the energy $E(t)$ of the system described by $L(z,\dot{z})$ for given initial conditions $z(t_0)$ and $\dot{z}(t_0)$? Illustrate graphically the trajectories of the system in the phasespace for various (real and complex) initial conditions and values of $\Omega_1$ and $\Omega_2$.

(d) Is there a simple way to derive the Lagrangian given above?

(e) Consider the case $\Omega_1 = -\,\Omega_2 = \Omega$. How does now the Lagrangian and the Lagrange equations of motion look like. Repeat (a), (b), (c) in this case. What happens to the explicit time-dependence of the Lagrangian? What has to be the sign of the imaginary part of $\Omega$ to achieve retarded or advanced motion?

(f) Consider the case $\Omega_1 = i\beta + \sqrt{\omega_0^2 - \beta^2}$ and $\Omega_2 = i\beta - \sqrt{\omega_0^2-\beta^2}$. How does now the Lagrangian and the Lagrange equations of motion look like. Repeat (a), (b), (c) in this case. What is the interpretation of $\beta$ for real $\beta$ and $\omega_0$?

(g) Are there examples of systems (a), (e), (f) in nature?

A conventional system described by one (complex) frequency $\Omega$ may be determined by the equation of motion $\dot{z} = -i\,\Omega\,z = \partial H(z,p)/\partial p$. The Hamiltonian of the system is therefore given by $H(z,p) = -\,i\,\Omega\,p\,z + f(z)$, while $f(z)$ is an arbitray function of $z$. Solve the Hamilton equation of motion $\dot{p} = -\,\partial H(z,p)/\partial z$ by the Green-function method. Show that the Lagrangian of the system may be expressed as:

$$L(z,\dot{z}) = [\dot{z} + i\,\Omega\,z][\int_{-\infty}^{t} dt' - \int_{t}^{\infty} dt'\,]\; e^{i\,\Omega\,(t-t')}\left[-\,\frac{\partial f(z(t'))}{\partial z(t')}\right] - f(z)\,.$$

Are the Lagrangian and the Hamiltonian related by a Legendre transform? Determine the Lagrange equation of motion corresponding to this Lagrangian. Discuss the result!