On the Consistency/Renormalizability of Relativistic Quantum Field Theories

6) Complex Lorentz-boost for arbitrary metric:

Let $g_{\mu\nu} = g^{\mu\nu}$ ($\mu, \nu \in \{0,1,2,3\}$) be an *arbitrary* complex symmetric metric tensor and its inverse. Furthermore define n^μ to be a time-like unit 4-vector, i.e. $n^2 = 1$, and ξ^μ a complex 4-vector like in problem 3). As in problem 3) we consider a Lorentz-transformation $\Lambda^\mu{}_\nu(\xi)$ which transforms the 4-vector $n^\mu \sqrt{\xi^2}$ into the 4-vector ξ^μ, i.e. $\xi^\mu = \Lambda^\mu{}_\nu(\xi)\, n^\nu \sqrt{\xi^2}$ and $n_\nu \sqrt{\xi^2} = \xi_\mu \Lambda^\mu{}_\nu(\xi)$. Show that the (symmetric) ansatz

$$\Lambda^\mu{}_\nu(\xi) \;=\; A\,\xi^\mu \xi_\nu + B\, n^\mu n_\nu + C\,(\xi^\mu n_\nu + n^\mu \xi_\nu) + D\, g^\mu{}_\nu$$

results in the following general two complex boost matrices ($\xi \cdot n := \xi^\mu n_\mu$):

$$\Lambda^\mu{}_\nu(\xi) \;=\; \pm \left\{ g^\mu{}_\nu - \frac{\sqrt{\xi^2}}{\sqrt{\xi^2} \mp \xi \cdot n} \left[n^\mu \mp \frac{\xi^\mu}{\sqrt{\xi^2}} \right] \left[n_\nu \mp \frac{\xi_\nu}{\sqrt{\xi^2}} \right] \right\}$$

Verify that this result is in fact a boost matrix and that it has the desired boost properties! Discuss the meaning of the two solutions! Verify that the boost matrix of problem 3) results in fact from the used metric (hint: apply the reflexion matrix $P^\mu_{\ \nu} := 2\, n^\mu n_\nu - g^\mu{}_\nu$)! Solve problem 3 (d) for several complex metrics of your taste! Illustrate resulting 4-momentum-hypersurfaces defined by the mass-shell condition and respective space-time trajectories! (For literature with respect to unconventional Lorentz-transformations see e.g. R.I. Sutherland, J.R. Shepanski, Phys. Rev. **D 33** (1986) 2896) or A. Brotas (IST), "An extension of the Lorentz Group" (preprint)).

7) The (anti-)causal Harmonic Oscillator (HO):

The Hamilton operator of the (anti-)causal HO is given by (ω = complex frequency):

$$H \;=\; H_{ret} + H_{adv} \;=\; \frac{\hbar\omega}{2}\,[c^+, a]_\pm \;+\; \frac{\hbar\omega^*}{2}\,[a^+, c]_\pm$$

with the non-vanishing (anti-)commutation relations $[a, c^+]_\mp = [c, a^+]_\mp = 1$ for Bosons/Fermions, respectively. Define the states $|n, m> := \frac{1}{\sqrt{n!\,m!}}\,(c^+)^n (a^+)^m |0>$ and $<n, m| := \frac{1}{\sqrt{n!\,m!}} <0|\, c^m\, a^n$, the density matrices $\rho := \exp(-\beta H)$, $\rho_{ret} := \exp(-\beta H_{ret})$, $\rho_{adv} := \exp(-\beta H_{adv})$ and their traces $Z := \mathrm{tr}\,\rho$, $Z_{ret} := \mathrm{tr}\,\rho_{ret}$ and $Z_{adv} := \mathrm{tr}\,\rho_{adv}$.

(a) Show that $H = H^+$, $H_{ret} = (H_{adv})^+$, $[H_{ret}, H_{adv}]_- = 0$, $< n, m|n', m' >= \delta_{nn'}\,\delta_{mm'}$ and that $< n, m|$ and $|n, m >$ are the left- and right-eigenvectors of H. What are the eigenvalues E_{nm} of H? What are the corresponding left- and right-eigenstates of H_{ret} and H_{adv} and the corresponding eigenvalues? Determine ρ in terms of ρ_{ret} and ρ_{adv}! Determine Z in terms of Z_{ret}, Z_{adv} or ρ_{ret}, ρ_{adv} or E_{nm}! How to determine the free energy and the entropy with respect to H, H_{ret}, and H_{adv}? How to introduce the concept of a chemical potential?

(b) Expand the solution $|\psi(t) >$ of the Schrödinger equation $i\hbar \frac{d}{dt} |\psi(t) >= H|\psi(t) >$ by its stationary solutions! What is the (adjoint) Schrödinger equation for $<\psi(t)|$?

(c) The space representation of the Schrödinger equation of the (anti-)causal HO is given by $i\hbar \frac{d}{dt} < z, z^*|\psi(t) >= \int dz' dz'^* < z, z^*|H|z', z'^* >< z', z'^*|\psi(t) >$ with

$$< z, z^*|H_{ret}|z', z'^* >= \left(-\frac{\hbar^2}{2\,M} \frac{d^2}{dz^2} + \frac{1}{2}\, M\, \omega^2\, z^2 \right) < z, z^*|z', z'^* > \;.$$

Determine $< z, z^*|A|z', z'^* >$ with $A \in \{H_{adv}, H, c^+, a^+, c, a, \rho\}$ and $< z, z^*|n, m >$, $< n, m|z', z'^* >$. How are $< z, z^*|n, m >$ and $< n, m|z', z'^* >$ normalized?

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8) The (anti-)causal Klein-Gordon field:

The Lagrange-density of the free (anti-)causal (Bosonic) Klein-Gordon field goes back to N. Nakanishi, Prog. Theor. Phys. Suppl. **51** (1972) 1 (and references therein). For an isospin $(N-1)/2$-field it is given by ($M = m - \frac{i}{2}\Gamma$):

$$\mathcal{L}_0(x) = \sum_{r=1}^{N} \left\{ \frac{1}{2}\left[(\partial\phi_r(x))^2 - M^2\,(\phi_r(x))^2 \right] + \frac{1}{2}\left[(\partial\phi_r^+(x))^2 - M^{*\,2}\,(\phi_r^+(x))^2 \right] \right\}$$

(a) Derive the Lagrange equations of motion for $\phi_r(x)$ and $\phi_r^+(x)$! The Lagrange equations of motion can be solved by a Laplace-transform. In symbolic notation we can write this transform as

$$\phi_r(x) := \phi_r(x, x_0) = \int \frac{d^4p}{(2\,\pi)^3}\,\delta(p^2 - M^2)\; e^{-i\,p\cdot(x-x_0)}\; a_r(p, x_0)$$

$$\phi_r^+(x) := \phi_r^+(x, x_0) = \int \frac{d^4p}{(2\,\pi)^3}\,\delta(p^2 - M^{*\,2})\; e^{+i\,p\cdot(x-x_0)}\; a_r^+(p, x_0)$$

Discuss the meaning of the symbolic complex δ-function in terms of Cauchy's theorem or the residuum sentence in the case, when $a_r(p, x_0)$ and $a_r^+(p, x_0)$ are strictly analytic! Show, that the expressions can be reformulated by ($\omega(|\vec{p}\,|) := \sqrt{\vec{p}^{\,2} + M^2}$ with $\omega(\,|\vec{0}\,|) = +M$)

$$\phi_r(x, x_0) = \int \frac{d^3p}{(2\,\pi)^3\,2\,\omega(|\vec{p}\,|)} \left[e^{-i\,p\cdot(x-x_0)}\; a_r(\vec{p}\,) + e^{+i\,p\cdot(x-x_0)}\; c_r^+(\vec{p}\,) \right]\Big|_{p^0=\omega(|\vec{p}\,|)}$$

$$\phi_r^+(x, x_0) = \int \frac{d^3p}{(2\,\pi)^3\,2\,\omega^*(|\vec{p}\,|)} \left[e^{-i\,p^*\cdot(x-x_0)}\; c_r(\vec{p}\,) + e^{+i\,p^*\cdot(x-x_0)}\; a_r^+(\vec{p}\,) \right]\Big|_{p^0=\omega(|\vec{p}\,|)}$$

with $a_r(\vec{p}\,) := a_r(p, x_0)|_{p^0=\omega(|\vec{p}\,|)}$ and $c_r^+(\vec{p}\,) := a_r(-p, x_0)|_{p^0=\omega(|\vec{p}\,|)}$. Derive from the equal-time commutation relations between fields and canonical conjugate momenta the commutation relations for $a_r(\vec{p}\,)$, $c_r(\vec{p}\,)$, $a_r^+(\vec{p}\,)$ and $c_r^+(\vec{p}\,)$! Derive the Hamiltonian of the system and compare it to the (anti-)causal HO of problem 7) !

(b) Define the vacuum $|0>$ with the properties $a|0>= c|0>= 0$ and $<0|a^+ = <0|c^+ = 0$! The time-ordering product for Bosons is given by $T[\phi(x)\phi(y)] := \theta(x^0 - y^0)\;\phi(x)\;\phi(y) \;+\; \theta(y^0 - x^0)\;\phi(y)\;\phi(x)$. Calculate — by using the standard integral representation of the θ-function — the propagator of the complex Klein-Gordon field with a complex mass $M = m - i\,\frac{\Gamma}{2}$, i.e. verify the relation $<0|T[\phi_r(x)\phi_s(y)]|0> = i\int \frac{d^4p}{(2\,\pi)^4}\,\frac{e^{-\,i\,p\cdot(x-y)}}{(p^2 - M^2)}\;\delta_{rs}$.

(c) Discuss now the case $N = 2$! Define the fields $\phi_\pm(x) := (\phi_1(x) \pm i\,\phi_2(x))/\sqrt{2}$. Reformulate the Lagrangian in terms of these fields! Show that there holds the continuity equation $\partial_\mu(\phi_+(x)\,\partial^\mu\phi_-(x) - \phi_-(x)\,\partial^\mu\phi_+(x)) = 0$! Is there a similar continuity equation in the case $N = 1$, i.e. for a neutral field? The Lagrangian is invariant under the global U(1)-gauge transformations $\phi_\pm(x) \to \exp(\pm\,i\,\alpha)\,\phi_\pm(x)$. What is the meaning of gauge invariance in terms of the fields $\phi_1(x)$ and $\phi_2(x)$?

9) Isospin and (global) gauge invariance in Quantum Mechanics (QM):

Assume that there are N states $\psi_r(t, z) = <z|\psi_r(t)>$ $(r = 1, \ldots, N)$ solving the same retarded Schrödinger equation $i\hbar\,\frac{d}{dt}\,\psi_r(t, z) = \left(-\frac{\hbar^2}{2\,M}\,\frac{d^2}{dz^2} + V(z)\right)\,\psi_r(t, z)$. Define for $N = 2$ the charged wavefunctions $\psi_\pm(t, z) := (\psi_1(t, z) \pm i\,\psi_2(t, z))/\sqrt{2}$! Solve problem 8 (c) for given wavefunctions by inspection of the Schrödinger equation without derivation of a Lagrangian! Why can't $|\psi(t, z)|^2$ be interpreted as a probability density of causal QM and why is the traditional normalization condition $\int dz|\psi(t, z)|^2 = 1$ inconsistent?