On the Consistency/Renormalizability of Relativistic Quantum Field Theories

10) Generalized Dirac spinors:

Let $u(p,s)$ $(s = \pm 1/2)$ be a generalized Dirac spinor fulfilling the generalized Dirac equation $(\not{p} - \sqrt{p^2}\,)\, u(p,s) = 0$. Define also the spinors $u^c(p,s) := C\,\gamma_0\, u^*(p,s)$ and $v(p,s) := u(-p,s)$. Using the notation of problem 6) the 4-vector p^μ is related to the 4-vector $\sqrt{p^2}\; n^\mu$ (with $n^2 = 1$) defining its rest frame by the complex boost $p^\mu = \Lambda^\mu{}_\nu(p)\;\sqrt{p^2}\; n^\nu$. The spinor in the rest frame fulfills of course $\sqrt{p^2}\,(\not{n} - 1\,)\, u(\sqrt{p^2}\; n, s) = 0$. Let χ_s be the standard 2-spinors with $\chi_s^+ \chi_{s'} = \delta_{ss'}$ and $\sum\limits_s \chi_s\,\chi_s^+ = 1_2$ $(s = \pm 1/2)$.

(a) What Dirac equations are fulfilled by the spinors $v(p,s)$, $u^c(p,s)$, $v^c(p,s)$, $\bar{u}(p,s)$, $\bar{v}(p,s)$, $\overline{u^c}(p,s)$ and $\overline{v^c}(p,s)$?

(b) Prove the analyticity property $u^c(p,s) = u(-p^*,s) = v(p^*,s)$!

(c) Lorentz covariance of the Dirac equation yields $S^{-1}(\Lambda(p))\,\gamma^\mu\, S(\Lambda(p)) = \Lambda^\mu{}_\nu(p)\,\gamma^\nu$. Show that $u(p,s) = S(p)\; u(\sqrt{p^2}\; n, s)$. Determine the matrix $S(p)$ for arbitrary metric $g_{\mu\nu}$ and for the metric of problem 3).

(d) Show for the metric of problem 3) that

$$u(p,s)|_{\mathrm{Re}[p^0]\,>\,0} \;=\; \frac{\not{p} + \sqrt{p^2}}{\sqrt{2\sqrt{p^2}\;(p^0 + \sqrt{p^2})}}\; u(\sqrt{p^2}, \vec{0}, s)\,.$$

Show that we may define $u(\sqrt{p^2},\vec{0},s)|_{\mathrm{Re}[p^0]\,>\,0} := \sqrt{2\sqrt{p^2}}\begin{pmatrix}\chi_s \\ 0\end{pmatrix}$! Determine the spinors $v(p,s)$, $u^c(p,s)$, $v^c(p,s)$, $\bar{u}(p,s)$, $\bar{v}(p,s)$, $\overline{u^c}(p,s)$ and $\overline{v^c}(p,s)$ for $\mathrm{Re}[p^0] > 0$ as a function of the corresponding spinors in the rest frame. What are the corresponding spinors in the rest frame? Show that there holds the normalization condition $\mathrm{sgn}[\,\mathrm{Re}(p^0)\,] \sum\limits_s u(p,s)\;\overline{v^c}(p,s) = \not{p} + \sqrt{p^2}$! Let $s^\mu(p) = \Lambda^\mu_\nu(p)\; s^\nu(\sqrt{p^2}\,,\vec{0}\,)$ and $s^\mu(\sqrt{p^2},\vec{0}\,) = (0,\vec{s}\,)$ with $|\vec{s}\,|^2 = 1$. Determine $s^\mu(p)\, s_\mu(p)$ and $p^\mu\, s_\mu(p)$! The spin projection s is defined by $\gamma_5\;\not{s}(p)\; u(p,s) = 2\,s\; u(p,s)$ for $\mathrm{Re}[p^0] > 0$. What are the respective equations for χ_s, $v(p,s)$, $u^c(p,s)$, $v^c(p,s)$?

(e) Define now for the metric of problem 3) the spinors on their "positive" mass shell, i.e. $u(\vec{p},s) := u(p,s)|_{p^0=\omega(|\vec{p}\,|)}$ and $v(\vec{p},s) := v(p,s)|_{p^0=\omega(|\vec{p}\,|)}$ with $\omega(|\vec{p}\,|) := \sqrt{\vec{p}^{\,2} + M^2}$ such that $\omega(|\vec{0}\,|) := M = m - i\,\frac{\Gamma}{2}$, and repeat for these spinors problems a), b), c) and d)!

11) Transposition properties of products formed by two Grassmann 4-spinors:

Let ψ_1 and ψ_2 be two Grassmann 4-spinors and $\overline{\psi_1^c} := \psi_1^T C$, $\overline{\psi_2^c} := \psi_2^T C$, $\overleftrightarrow{\partial}_\mu := \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu$. Show the following transposition identities:

$$\begin{aligned}
(\,\overline{\psi_1^c}\; 1_4\; \psi_2\,)^T &= \overline{\psi_2^c}\; 1_4\; \psi_1 \\
(\,\overline{\psi_1^c}\; \gamma^5\; \psi_2\,)^T &= \overline{\psi_2^c}\; \gamma^5\; \psi_1 \\
(\,\overline{\psi_1^c}\; \gamma^\mu\; \psi_2\,)^T &= -\,\overline{\psi_2^c}\; \gamma^\mu\; \psi_1 \\
(\,\overline{\psi_1^c}\; \gamma^\mu\gamma^5\; \psi_2\,)^T &= \overline{\psi_2^c}\; \gamma^\mu\gamma^5\; \psi_1 \\
(\,\overline{\psi_1^c}\; \sigma^{\mu\nu}\; \psi_2\,)^T &= -\,\overline{\psi_2^c}\; \sigma^{\mu\nu}\; \psi_1 \\
(\,\overline{\psi_1^c}\; \gamma^\mu\, \overleftrightarrow{\partial}_\mu\;\; \psi_2\,)^T &= \overline{\psi_2^c}\; \gamma^\mu\, \overleftrightarrow{\partial}_\mu\;\; \psi_1 \\
(\,\overline{\psi_1^c}\; \gamma^\mu\gamma^5\, \overleftrightarrow{\partial}_\mu\;\; \psi_2\,)^T &= -\,\overline{\psi_2^c}\; \gamma^\mu\gamma^5\, \overleftrightarrow{\partial}_\mu\;\; \psi_1
\end{aligned}$$

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12) Lagrangian of spin 1/2 Fermion minimally coupled to Abelian spin 1 Boson:

The Lagrange-density of the Fermionic sector of (anti-)causal isospin 1/2 Quantum Electrodynamics (QED) is given by ($\psi_\pm(x) := (\psi_1(x) \pm i\,\psi_2(x))/\sqrt{2}$ and $M := m - i\,\frac{\Gamma}{2}$):

$$\begin{aligned}\mathcal{L}_\psi(x) &= \frac{1}{2}\left(\overline{\psi_1^c}(x), \overline{\psi_2^c}(x)\right)\begin{pmatrix}\left[\frac{i}{2}\gamma^\mu \overleftrightarrow{\partial}_\mu - M\right] & -i\,e\,\not{A} \\ +i\,e\,\not{A} & \left[\frac{i}{2}\gamma^\mu \overleftrightarrow{\partial}_\mu - M\right]\end{pmatrix}\begin{pmatrix}\psi_1(x) \\ \psi_2(x)\end{pmatrix} + \text{h.c.} \\ &= \overline{\psi_+^c}(x)\,(\frac{i}{2}\,\overleftrightarrow{\not{\partial}} + e\,\not{A} - M)\,\psi_-(x) \;+\; \overline{\psi_-}(x)\,(\frac{i}{2}\,\overleftrightarrow{\not{\partial}} + e^*\,\gamma^\mu A_\mu^+ - \gamma_0 M^+\gamma_0)\,\psi_+^c(x)\end{aligned}$$

(a) Show that the gauge transformation yielding $e\,\not{A}' = e\,\not{A} + [\,\not{\partial}, \Lambda(x)]$ is obtained by $\psi_\pm'(x) = \exp(\mp i\,\Lambda(x))\;\psi_\pm(x)$ or equivalently

$$\begin{pmatrix}\psi_1'(x) \\ \psi_2'(x)\end{pmatrix} = \begin{pmatrix}\cos\Lambda(x) & \sin\Lambda(x) \\ -\sin\Lambda(x) & \cos\Lambda(x)\end{pmatrix}\begin{pmatrix}\psi_1(x) \\ \psi_2(x)\end{pmatrix}\,.$$

(b) Try to construct as many independent Noether-identities with respect to the Lagrangian under consideration. What are the resulting conserved and non-conserved currents and their continuity equations in the case $A \neq 0$ and $A = 0$? What are the symmetry transformations and the continuity equations related to the currents $\overline{\psi_+^c}(x)\,\gamma^\mu\,\psi_-(x)$, $\overline{\psi_+^c}(x)\,\gamma^\mu\gamma_5\,\psi_-(x)$ and $\overline{\psi_+}(x)\,\gamma^\mu\gamma_5\,\psi_-(x)$? Discuss the limit $M \to -i\,\varepsilon$ in the case $A = 0$. Is "chiral symmetry" really a symmetry? Is there a conserved current for $M \to -i\,\varepsilon$ and $A = 0$? What is the meaning of the underlying symmetry of such a current?

(c) What will happen in the isospin 0 case, i.e. in the case $\psi_2(x) = 0$, which applies e.g. to (anti-)causal massive or massless Neutrinos? Are there still global or local symmetries in the Lagrangian? Is there any way to couple gauge-fields with arbitrary Dirac-structure minimally to this Lagrangian of chargeless spin 1/2 Fermions?

13) Spin 1/2 Fermions minimally coupled to non-Abelian spin 1 Bosons (I):

There are many possible ways to generalize the Lagrangian of problem 12) and related gauge transformations to the non-Abelian case. Try to find as many scenarios as possible! Consider the case of the gauge group SU(N) with $N = 2$ and 3! Use the fact, that some generators λ^a ($a = 1, \ldots N^2 - 1$) of the gauge group are symmetric, i.e. $(\lambda_S^a)^T = \lambda_S^a$, and some are antisymmetric, i.e. $(\lambda_A^a)^T = -\lambda_A^a$, i.e. the sum of the products of gauge fields with generators of the gauge group can be split up into a sum of symmetric and antisymmetric parts, i.e. $A^a\lambda^a = A_S^a\lambda_S^a + A_A^a\lambda_A^a$. Consider the conserved and broken Noether symmetries of the Lagrangians you construct!