On the Consistency/Renormalizability of Relativistic Quantum Field Theories

14) **Norm & probability current densitiy of (un-)charged wavefunctions/fields:**
As introduced in the lecture the (neutral) retarded Schrödinger equation with a time-dependent interaction $V(t,\vec{x}\,)$ and the corresponding free Klein-Gordon and the Dirac equation and their adjoints are ($M = m - \frac{i}{2}\Gamma$) ($\hbar = c = 1$):

$$\begin{aligned}
i\,\frac{d}{dt} <\vec{x}\,|\psi(t)> &= \left(-\frac{1}{2\,M}\,\vec{\nabla}^{\,2} + V(t,\vec{x}\,)\right) <\vec{x}\,|\psi(t)>\,,\\
-\,i\,<\psi(t)|\vec{x}>\,\overleftarrow{\frac{d}{dt}} &= <\psi(t)|\vec{x}>\left(-\frac{1}{2\,M}\,\overleftarrow{\nabla}^{\,2} + V(t,\vec{x}\,)\right)\,,\\
(-\,\partial^2 - M^2)\,\phi^{(\pm)}(x) &= 0 \quad , \quad \phi^{(\pm)}(x)\,(-\,\overleftarrow{\partial}^{\,2} - M^2) \;=\; 0\,,\\
(i\,\partial\!\!\!/ - M)\,\psi^{(\pm)}(x) &= 0 \quad , \quad \overline{\psi^{(\pm)\,c}}(x)\,(-\,i\,\overleftarrow{\partial\!\!\!/} - M) \;=\; 0\,.
\end{aligned}$$

The advanced equations are obtained by Hermitian conjugation. Here $\phi^{(\pm)}(x)$ and $\psi^{(\pm)}(x)$ are the positive (+) and negative (−) frequency part of the Klein-Gordon field $\phi(x)$ and the Dirac field $\psi(x)$, i.e. ($\omega(|\vec{p}\,|) := \sqrt{\vec{p}^{\,2} + M^2}$ with $\omega(\,|\vec{0}\,|) = +M$):

$$\begin{aligned}
\phi(x) &= \phi^{(+)}(x) + \phi^{(-)}(x) = \int \frac{d^3p}{(2\,\pi)^3\,2\,\omega(|\vec{p}\,|)}\,\Big[\,e^{-i\,p\cdot x}\,a(p) + e^{+i\,p\cdot x}\,a(-p)\,\Big]\Big|_{p^0=\omega(|\vec{p}\,|)}\\
\psi(x) &= \psi^{(+)}(x) + \psi^{(-)}(x)\\
&= \sum_s \int \frac{d^3p}{(2\,\pi)^3\,2\,\omega(|\vec{p}\,|)}\,\Big[\,e^{-i\,p\cdot x}\,b(p,s)\,u(p,s) + e^{+i\,p\cdot x}\,b(-p,s)\,v(p,s)\,\Big]\Big|_{p^0=\omega(|\vec{p}\,|)}
\end{aligned}$$

According to the lecture we can compactify the notation by defining $a(\vec{p}\,) := a(p)|_{p^0\,=\,\omega(|\vec{p}\,|)}$, $c^+(\vec{p}\,) := a(-p)|_{p^0\,=\,\omega(|\vec{p}\,|)}$, $b(\vec{p},s) := b(p,s)|_{p^0\,=\,\omega(|\vec{p}\,|)}$ and $d^+(\vec{p},s) := b(-p,s)|_{p^0\,=\,\omega(|\vec{p}\,|)}$.

(a) Derive from the Schödinger, Klein-Gordon, Dirac equations (& adjoints) the following retarded continuity equations ($\psi(t,\vec{x}\,) :=<\vec{x}\,|\psi(t)>$, $\tilde{\psi}(t,\vec{x}\,) :=<\psi(t)|\vec{x}>$):

$$\begin{aligned}
\frac{d}{dt}\left(\tilde{\psi}(t,\vec{x}\,)\,\psi(t,\vec{x}\,)\right) &= \vec{\nabla}\cdot\left[\frac{i}{2\,M}\left(\tilde{\psi}(t,\vec{x}\,)\left(\vec{\nabla}\,\psi(t,\vec{x}\,)\right) - \left(\vec{\nabla}\,\tilde{\psi}(t,\vec{x}\,)\right)\psi(t,\vec{x}\,)\right)\right]\\
0 &= \partial_\mu\left[-\,\phi^{(\mp)}(x)\left(\partial^\mu\,\phi^{(\pm)}(x)\right) + \left(\partial^\mu\,\phi^{(\mp)}(x)\right)\phi^{(\pm)}(x)\right]\\
0 &= \partial_\mu\left[\,i\,\overline{\psi^{(\mp)\,c}}(x)\,\gamma^\mu\,\psi^{(\pm)}(x)\,\right]
\end{aligned}$$

(b) What are the corresponding advanced continuity equations?

(c) What is the Noether symmetry which leads to the continuity equations $\partial_\mu\,j^\mu(x) = 0$? What is the interpretation of the Noether current density 4-vector $j^\mu(x)$?

(d) It is possible to identify $j^{\,0}(x)$ with the probability density, $\int d^3x\,j^{\,0}(x)$ with the norm and $\vec{j}(x)$ with the probability current density 3-vector. Identify these quantities for the retarded and advanced Schrödinger, Klein-Gordon and Dirac equations! Why can't $|\psi(t,\vec{x}\,)|^2$ be identified with the probability density of the Schrödinger equation and $\int d^3x\,|\psi(t,\vec{x}\,)|^2$ with the corresponding norm?

(e) Generalize the continuity equations to finite isospin, i.e. by introducing N fields $\psi_r(t,\vec{x}\,)$, $\phi_r(x)$ or $\psi_r(x)$ with $r = 1,\ldots,N$. Study how for $N = 1$ and 2 the probability current densities enter the charge (isospin) current density and why for $N = 0$ the charge current density vanishes!

(f) Introduce interaction terms to the Klein-Gordon and Dirac equations! Study under what circumstances the probability current densities maintain to be conserved!

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15) Spin 1/2 Fermions minimally coupled to non-Abelian spin 1 Bosons (II):

The Lagrange-density of the Fermionic sector of (anti-)causal isospin 1/2 Quantum Chromodynamics (QCD) with one flavour is given by ($\psi_\pm(x) := (\psi_1(x) \pm i\,\psi_2(x))/\sqrt{2}$, $M := m - i\,\frac{\Gamma}{2}$, $A^\mu = A^{a\,\mu}\,\lambda^a/2$, $A^\mu_S := (A^\mu + (A^\mu)^T)/2$, $A^\mu_A := (A^\mu - (A^\mu)^T)/2$):

$$\begin{aligned}\mathcal{L}^{QCD}_\psi(x) &= \frac{1}{2}\left(\overline{\psi^c_1}(x), \overline{\psi^c_2}(x)\right)\begin{pmatrix}\left[\frac{i}{2}\gamma^\mu \overset{\leftrightarrow}{\partial}_\mu + g\,\not{A}_A - M\right] & -i\,g\,\not{A}_S \\ +i\,g\,\not{A}_S & \left[\frac{i}{2}\gamma^\mu \overset{\leftrightarrow}{\partial}_\mu + g\,\not{A}_A - M\right]\end{pmatrix}\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} + \text{h.c.} \\ &= \overline{\psi^c_+}(x)\,(\,\frac{i}{2}\,\overset{\leftrightarrow}{\not\partial} + g\,\not{A} - M)\,\psi_-(x) \;+\; \overline{\psi_-}(x)\,(\,\frac{i}{2}\,\overset{\leftrightarrow}{\not\partial} + g^*\,\gamma^\mu A^+_\mu - \gamma_0 M^+\gamma_0)\,\psi^c_+(x)\end{aligned}$$

(a) Show that the infinitesimal gauge transformation yielding $g\,\not{A}' = g\,\not{A} + [\,\not\partial, \Lambda(x)]$ is obtained by $\psi'_-(x) = \exp(+i\,\Lambda(x))\;\psi_-(x)$ and $\psi'_+(x) = \exp(-i\,(\Lambda(x))^T)\;\psi_+(x)$ with $\Lambda = \Lambda^a\,\lambda^a/2$, $\Lambda_S := (\Lambda + \Lambda^T)/2$, $\Lambda_A := (\Lambda - \Lambda^T)/2$. What is the corresponding gauge transformation, which transforms $\psi_1(x)$, $\psi_2(x)$ into $\psi'_1(x)$, $\psi'_2(x)$? Construct corresponding Noether identities and Noether currents!

(b) Decompose the Fermion fields into right- and left-handed fields, i.e. perform $\psi_\pm(x) = \psi_{\pm R}(x) + \psi_{\pm L}(x)$ with $\psi_{\pm R}(x) := P_R\,\psi_\pm(x)$, $\psi_{\pm L}(x) := P_L\,\psi_\pm(x)$ and $P_R := (1+\gamma_5)/2$, $P_L := (1-\gamma_5)/2$. Express $\mathcal{L}^{QCD}_\psi(x)$ in terms of these fields and discuss the result!

(c) Replace the interaction term in $\mathcal{L}^{QCD}_\psi(x)$, i.e. $\mathcal{L}_V(x) = \overline{\psi^c_+}(x)\,g\,\not{A}\,\psi_-(x) + \text{h.c.}$, by $\mathcal{L}_A(x) = \overline{\psi^c_+}(x)\,g\,\not{A}\gamma_5\,\psi_-(x) + \text{h.c.}$ or linear combinations like $(\mathcal{L}_V(x) \pm \mathcal{L}_A(x))/2$ and try to discuss the (gauge) symmetry properties of the resulting Lagrange densities. Is it possible to construct a gauge invariant (anti-)causal Fermionic part of a Lagrangian which may serve as a prototype of the standard model of elementary particle physics with parity violating intermediate vector Bosons $Z, W^\pm$ and massive Fermions? Why do the Fermion mass terms cause so much troubles?

(d) Perform the limit of $\mathcal{L}^{QCD}_\psi(x)$ to uncharged QCD, i.e. set $\psi_2(x) = 0$. Identify the remaining gauge symmetry and gauge fields starting from gauge groups SU(2) and SU(3)! Compare isospin 1/2 QED and isospin 0 QCD obtained from SU(2)!

16) Vacuum polarization of QED with complex Fermion masses:

The one loop vacuum polarization of QED (see also problem 12!) in d-dimensional (Minkowski) momentum space for loop Fermions with complex mass $M = m - \frac{i}{2}\,\Gamma$ is given by:

$$\Pi^{\mu\nu}(q) \;=\; -\,i\,e^2\int\frac{d^dp}{(2\pi)^d}\,\frac{\text{tr}[\,\gamma^\mu(\,\not{p} + M)\gamma^\nu(\,\not{p} - \not{q} + M)\,]}{(p^2 - M^2)((p-q)^2 - M^2)}$$

We know that for $d = 3+1$ the integral is — due to gauge invariance — for $\Gamma \to \varepsilon = +0$ not quadratically divergent, as expected by naive power counting, yet merely logarithmically divergent.

(a) Argue by power counting why a finite value for the imaginary part Γ of the (complex) Fermion mass in $d = 3+1$ dimensions will not change the logarithmic divergence of the integral, yet will yield only finite corrections to the integral! Calculate these finite corrections!

(b) Evaluate the integral for $d = 2+1$ or $d = 1+1$ using the Feynman integral method and Wick rotation! Discuss the expected finite result as a function of Γ!