Dyson-Schwinger Techniques and the Bethe-Salpeter Equation (Introduction)

1) $\varphi^2\,\chi$-model:

Starting point is the classical action $S^{(0)} = \int d^4x\ \mathcal{L}^{(0)}(x)$ of three interacting scalar Bosons φ_1, φ_2 and $\varphi_3 \stackrel{!}{=} \chi$ with masses m_1, m_2 and $m_3 \stackrel{!}{=} \mu$, respectively. Within the $\varphi^2\,\chi$-model the corresponding Lagrange-density is given by ($g_{\mu\nu} = \text{Diag}(+,-,-,-)$):

$$\begin{aligned}\mathcal{L}^{(0)}(x) &= \frac{1}{2}\Big((\partial_x\,\varphi_1(x))^2 - m_1^2\,\varphi_1^2(x)\Big) + \frac{1}{2}\Big((\partial_x\,\varphi_2(x))^2 - m_2^2\,\varphi_2^2(x)\Big)\\ &+ \frac{1}{2}\Big((\partial_x\,\chi(x))^2 - \mu^2\,\chi^2(x)\Big) - \frac{g}{\sqrt{2}}\Big(\varphi_1^2(x)\,\chi(x) + \varphi_2^2(x)\,\chi(x)\Big)\ .\end{aligned}$$

After the definition of

$$Z[j_1,j_2,j_3] = \int D\varphi_1 D\varphi_2 D\varphi_3\ \exp\left[\frac{i}{\hbar}\Big(S^{(0)}[\varphi_1,\varphi_2,\varphi_3] + \sum_{\ell=1,2,3}\int d^4x\ \varphi_\ell(x)\,j_\ell(x)\Big)\right]$$

we can introduce the generating functionals $T\,[\vec{j}\,]$ and $W\,[\vec{j}\,]$ for ordinary and connected n-point Green's functions G_n and $G_{n\,c}$ by $T\,[\vec{j}\,] = \exp(\frac{i}{\hbar}W\,[\vec{j}\,]) = Z\,[\vec{j}\,]/Z\,[\vec{0}\,]$. The corresponding Green's functions are given by:

$$G_{n\,;\ell_1\ldots\ell_n}(x_1,\ldots,x_n) := \left(\frac{\hbar}{i}\right)^n \left.\frac{\delta^n\,T\,[\vec{j}\,]}{\delta j_{\ell_1}(x_1)\ldots\delta j_{\ell_n}(x_n)}\right|_{\vec{j}=\vec{0}} =: G_{n\,;\ell_1\ldots\ell_n}(x_1,\ldots,x_n;\vec{j}\,)\Big|_{\vec{j}=\vec{0}}$$

$$G_{n\,c\,;\ell_1\ldots\ell_n}(x_1,\ldots,x_n) := \left(\frac{\hbar}{i}\right)^n \left.\frac{\delta^n\,\frac{i}{\hbar}W\,[\vec{j}\,]}{\delta j_{\ell_1}(x_1)\ldots\delta j_{\ell_n}(x_n)}\right|_{\vec{j}=\vec{0}} =: G_{n\,c\,;\ell_1\ldots\ell_n}(x_1,\ldots,x_n;\vec{j}\,)\Big|_{\vec{j}=\vec{0}}$$

Perform now the following transformations:

(a) Determine $G_{n\,;\ell_1\ldots\ell_n}$ as a function of $G_{n\,c\,;\ell_1\ldots\ell_n}$ for $n = 1,2,3,4$ and $\ell_i = 1,2,3$.

(b) Determine the coupled system of Dyson-Schwinger equations for all G_n upto $n = 4$.

(c) Determine the coupled system of Dyson-Schwinger equations for $G_{n,c}$ upto $n = 4$.

(d) Determine from the systems (b) and (c) Dyson's (gap-)equations for G_2 and $G_{2\,c}$.

(e) Perform the limit $G_1 = G_{1c} \to 0$ (symmetric vacuum) and determine from the systems (b) and (c) the Bethe-Salpeter equation (BSE) for $G_{4\,;1,2,1,2}$ and $G_{4\,c\,;1,2,1,2}$.

(f) A typical inhomogeneous BSE takes the following form:

$$\begin{aligned}G_{4\,c}(x_1',x_2',x_1,x_2) &= G_{2\,c}(x_1',x_1)\ G_{2\,c}(x_2',x_2)\\ &+ \int d^4z_1\,d^4z_2\,d^4z_1'd^4z_2'\ G_{2\,c}(x_1',z_1)\ G_{2\,c}(x_2',z_2)\,K(z_1,z_2,z_1',z_2')\,G_{4\,c}(z_1',z_2',x_1,x_2)\ .\end{aligned}$$

Here $K(x_1,x_2,x_1',x_2')$ is the so called irreducible Bethe-Salpeter kernel. What would be the Bethe-Salpeter kernel for (e) in the case of a simple "ladder" exchange of a χ-Boson for $m_1 \neq m_2$ and $m_1 = m_2$?

(g) Fourier-transform (e) by $f(x) = \int \frac{d^4p}{(2\pi)^4}\,e^{ipx}f(p)$ and $f(x') = \int \frac{d^4p'}{(2\pi)^4}\,e^{-ip'x'}f(p')$.

(h) Perform a formal multiplicative renormalization of (1) and $(a),\ldots,(g)$. What has in general to be done to stabilize the $\varphi^2\,\chi$-model?

(i) Perform the left work as homework!

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[2] 1st course in "QCD and more", CFIF postgraduate lecture courses 2001

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2) Functional differentiation and integration for Grassmann-fields:

As already addressed in the lecture Bosonic fields can be described by commuting functions $\varphi_i(x)$ with $\varphi_i(x)\,\varphi_j(y) = +\varphi_j(y)\,\varphi_i(x)$. These functions can be treated like variables with respect to functional differentiation and functional integration ("path-integration"). Regular functionals $F[\varphi_1,\ldots,\varphi_n]$ can be Taylor expanded in standard manner, i.e.:

$$F[\vec{\varphi}\,] \;=\; \sum_n \frac{1}{n!}\sum_{i_1}\ldots\sum_{i_n}\int d^4x_1\ldots d^4x_n\;\left.\frac{\delta^n F[\vec{\varphi}\,]}{\delta\varphi_{i_1}(x_1)\ldots\delta\varphi_{i_n}(x_n)}\right|_{\vec{\varphi}=\vec{0}}\varphi_{i_1}(x_1)\ldots\varphi_{i_n}(x_n)$$

Symbolically I just write $F[\vec{\varphi}\,] = \sum\limits_n \sum\limits_{i_1<\ldots<i_n} f^{(n)}_{i_1\ldots i_n}\,\varphi_{i_1}\ldots\varphi_{i_n}$. With respect to functional integration and differentiation of Bosonic fields the following relations hold in the symbolic notation:

(a) $\frac{\delta^2}{\delta\varphi_i\delta\varphi_j} = \frac{\delta^2}{\delta\varphi_j\delta\varphi_i}$, $\int D\varphi_i\,D\varphi_j = \int D\varphi_j D\varphi_i$, $\frac{\delta}{\delta\varphi_j}\,\varphi_i = \delta_{ij}$, $\frac{\delta}{\delta\varphi_j}\,\varphi_i\,\varphi_k = \delta_{ij}\,\varphi_k + \varphi_i\,\delta_{jk}$

(b) $\int D\varphi_n\ldots D\varphi_1(a\,F[\vec{\varphi}\,] + b\,G[\vec{\varphi}\,]) = a\int D\varphi_n\ldots D\varphi_1 F[\vec{\varphi}\,] + b\int D\varphi_n\ldots D\varphi_1\,G[\vec{\varphi}\,]$

(c) $\int D\varphi_n\ldots D\varphi_1\;F[\vec{\varphi}+\vec{\psi}\,] = \int D\varphi_n\ldots D\varphi_1 F[\vec{\varphi}\,]$

(d) $\int D\varphi'_n\ldots D\varphi'_1\;F[\vec{\varphi}\,'] = \int D\varphi_n\ldots D\varphi_1\;\det\left[\dfrac{\delta\vec{\varphi}\,'}{\delta\vec{\varphi}}\right]\;F[\vec{\varphi}\,(\vec{\varphi}\,')]$ for $\varphi'_i = \sum\limits_j A_{ij}\phi_j + \psi_i$.

(e) $\int D\varphi_n\ldots D\varphi_1\;\exp(-\sum\limits_{ij}\varphi_i\,A_{ij}\,\varphi_j) \propto (\det{}'A)^{-1/2}$

In order to obey the Pauli-principle for Fermionic fields one can introduce now anti-symmetic Grassmann variables by $\varphi_i(x)\,\varphi_j(y) = -\varphi_j(y)\,\varphi_i(x)$ (in symbolic notation $\varphi_i\,\varphi_j = -\varphi_j\,\varphi_i$). What will change for Grassmann fields in relations (a), …, (e)?

3) Numerical solution of the homogenous "ladder" BSE of the $\varphi^2\,\chi$-model (I):

As will be discussed at another place (see e.g. hep-ph/9509382) the *homogeneous* "ladder" BSE of problem 1 (f) in momentum space can be — after Wick-rotation to Euclidean coordinates — and expansion of the eigenvectors in 4-dimensional spherical harmonics simplified to an infinite-dimensional eigenvalue problem (eigenvalue λ) of the form (ℓ is the 3-dimensional spherical angular momentum quantum number):

$$g_{L,\ell}(z) \;=\; \lambda\sum_{L'=\ell}^{\infty}\int_{-1}^{1} dz'\;K_{LL',\ell}(z,z')\;g_{L',\ell}(z') \qquad (L=\ell,\ell+1,\ldots)$$

This coupled system of integral equations belongs to the class of Fredholm integral equations of the second type. It can be numerically solved by the *Quadrature Method.* Quadrature means, that the integral is discretised by $\int_a^b dz\,f(z) \to \sum\limits_{k=1}^{N} w_k f(z_k) + \varepsilon_{_N}[f]$, while z_k are the *quadrature interpolation nodes*, w_k are numerical coefficients independent of the choice of $f(z)$ and $\varepsilon_{_N}[f]$ is the reminder of the truncation error. For $\ell = 0$ we have (ignoring $\varepsilon_{_N}[f]$ and choosing $L' \le L_c$)(Typical values: $N \simeq 45$, $L_c \simeq 7$):

$$g_{L,0}(z_j) \;=\; \lambda\sum_{L'=0}^{L_c}\sum_{k=1}^{N} K_{LL',0}(z_j,z_k)\;w_k\;g_{L',0}(z_k) \qquad (L=0,\ldots,L_c,\quad j=1,2,\ldots,N)$$

Write a numerical code for the solution of such a discretised $N\cdot(L_c+1)$-dimensional eigenvalue problem using Gauss–Legendre quadrature interpolation nodes and corresponding weights w_k. The use of numerical routines like the NAG-package is welcome! Use complex variables, as complex eigenvalues are expected!.