Dyson-Schwinger Techniques and the Bethe-Salpeter Equation (Introduction)

4) Quasipotential (QP) reduction of the scalar BSE (I):

For the following steps it is useful to introduce Jacobi-coodinates for particles of mass m_1 and m_2 ($\eta_1 := m_1/(m_1+m_2)$, $\eta_2 := m_2/(m_1+m_2)$, $\eta_1+\eta_2 = 1$):

$$\begin{array}{rclcrcl} X & := & \eta_1\, x_1 \;+\; \eta_2\, x_2 & , & x & := & x_1 - x_2 \quad , \\ q & := & \eta_2\, p_1 \;-\; \eta_1\, p_2 & , & P & := & p_1 + p_2 \end{array}$$

(a) Show that the following identities hold: $p_1\, x_1 + p_2\, x_2 = P\, X + q\, x$; $\int d^4p_1\, d^4p_2 = \int d^4P\, d^4q$; $\int d^4x_1\, d^4x_2 = \int d^4X\, d^4x$; $\delta^4(p_1'-p_1)\;\delta^4(p_2'-p_2) = \delta^4(P'-P)\;\delta^4(q'-q)$.

The Fourier-transform of the Bethe-Salpeter of the BSE in problem 1 (f) is:

$$G_{4\,c}(p_1',p_2',p_1,p_2) \;=\; G_{2\,c}(p_1',p_1)\; G_{2\,c}(p_2',p_2)$$
$$+ \int \frac{d^4\bar{p}_1}{(2\pi)^4}\,\frac{d^4\bar{p}_2}{(2\pi)^4}\,\frac{d^4\bar{p}_1'}{(2\pi)^4}\,\frac{d^4\bar{p}_2'}{(2\pi)^4}\; G_{2\,c}(p_1',\bar{p}_1)\; G_{2\,c}(p_2',\bar{p}_2)\; K(\bar{p}_1,\bar{p}_2,\bar{p}_1',\bar{p}_2')\; G_{4\,c}(\bar{p}_1',\bar{p}_2',p_1,p_2)\ .$$

This equation can be significanty simplified using *translational invariance*, which yields:

$$\begin{array}{rcl} G_{2\,c}(p_j',p_j) & \stackrel{!}{=} & (2\pi)^4\,\delta^4(p_j'-p_j)\; i\,\Delta_F(p_j) \qquad (j=1,2) \\ G_{4\,c}(p_1',p_2',p_1,p_2) \;=:\; G_{4\,c}(P',q';P,q) & \stackrel{!}{=} & (2\pi)^4\,\delta^4(P'-P)\; G_{4\,c}(P;q',q) \\ K(p_1',p_2',p_1,p_2) \;=:\; K(P',q';P,q) & \stackrel{!}{=} & (2\pi)^4\,\delta^4(P'-P)\; K(P;q',q) \end{array}$$

(b) Show, that the Fourier-transformed BSE is reduced to ($S_{\text{scalar}}(P,q) := \Delta_F^{-1}(p_1)\Delta_F^{-1}(p_2)$):

$$S_{\text{scalar}}(P,q)\; G_{4\,c}(P;q,q') = (2\pi)^4\,\delta^4(q-q') \;-\; \int \frac{d^4\bar{q}}{(2\pi)^4}\; K(P;q,\bar{q})\; G_{4\,c}(P;\bar{q},q')\ .$$

For a *spherical symmetrical* interaction kernel one can introduce spherical harmonics, integrate out the angular dependences and reduce this *coupled system of four integral equations* to a system of *two coupled integral equations.* The idea of the QP approximation is now to reduce the BSE without changing the kernel to one — numerically easy handable — integral equation by introducing covariantly in $S_{\text{scalar}}(P,q)$ a δ-distribution:

$$S_{\text{scalar}}(P,q) \to S^{\text{QP}}_{\text{scalar}}(P,q) = 2\pi\, i \int_{(m_1+m_2)^2}^{\infty} ds'\; \frac{f(s',s)}{s' - s - i\,\varepsilon}\;\cdot$$
$$\cdot\; \delta^{(+)}\Big[\Big(\eta_1(s')\sqrt{\frac{s'}{s}}\,P + q\Big)^2 - m_1^2\Big]\; \delta^{(+)}\Big[\Big(\eta_2(s')\sqrt{\frac{s'}{s}}\,P - q\Big)^2 - m_2^2\Big] \qquad (s := P^2)$$

Here $f(s',s)$ is an arbitrary function with $f(s,s)=1$, and $\eta_1(s)$ and $\eta_2(s)$ are given by:

$$\eta_1(s) \;:=\; \frac{1}{2}\left(1+\frac{m_1^2-m_2^2}{2s}\right) \;\stackrel{!}{=}\; \frac{E_1^{cm}}{\sqrt{s}} \quad , \quad \eta_2(s) \;:=\; \frac{1}{2}\left(1-\frac{m_1^2-m_2^2}{2s}\right) \;\stackrel{!}{=}\; \frac{E_2^{cm}}{\sqrt{s}}$$

(c) Show that in the cm-frame the QP propagator reduces to ($\omega_j(|\vec{q}\,|) := \sqrt{\vec{q}^{\,2}+m_j^2}$):

$$S^{\text{QP}}_{\text{scalar}}(P_{cm}\,,q) \;\stackrel{!}{=}\; 2\pi\, i\; \frac{f((\omega_1(|\vec{q}\,|)+\omega_2(|\vec{q}\,|))^2,s)}{((\omega_1(|\vec{q}\,|)+\omega_2(|\vec{q}\,|))^2 - s)}\; \frac{(\omega_1(|\vec{q}\,|)+\omega_2(|\vec{q}\,|))}{2\,\omega_1(|\vec{q}\,|)\,\omega_2(|\vec{q}\,|)}\;\delta(q^0)$$

The QP framework can be reinterpreted in terms of a one-dimensional degree of freedom ("spurion") propagating in the presence of two bound onshell components (see e.g. R.J. Yaes, Phys. Rev. D3 (1971) 3086). The *spurion* guarantees *covariance, 4-momentum conservation and unitarity* within the infinitely many possible QP equations.

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5) Quasipotential (QP) reduction of the scalar BSE (II):

Starting point are the results of problem 4. Cooper and Jennings (1988, 1989 and 1993) pointed out, that not all infinitely possible QP equations yield for infinite mass of one component the correct one-particle limit (OPL), i.e. the Klein-Gordon equation for the other component. On the other hand they gave a prescription to construct (scalar) QP equations with the correct OPL by making the ansatz:

$$f(s',s) \quad = \quad \frac{4\sqrt{s\,s'}\,\eta_1(s')\,\eta_2(s')\;s'}{s\,s' - (m_1^2-m_2^2)^2}\;g(s',s) \quad \stackrel{!}{=} \quad \frac{\sqrt{s}\,(s'^2 - (m_1^2-m_2^2)^2)}{\sqrt{s'}\,(s\,s' - (m_1^2-m_2^2)^2)}\;g(s',s)$$

The new function $g(s',s)$ has to obey the same property $g(s,s) = 1$ as the original function $f(s',s)$.

(a) Show, that the $S^{\,\mathrm{QP}}_{\mathrm{scalar}}(P\,,q)$ in terms of $g(s',s)$ is given in the cm–frame by:

$$S^{\,\mathrm{QP}}_{\mathrm{scalar}}(P_{cm}\,,q) \;\stackrel{!}{=}\; 2\pi\,i\;\frac{g((\omega_1(|\vec{q}\,|)+\omega_2(|\vec{q}\,|))^2,s)}{((\omega_1(|\vec{q}\,|)+\omega_2(|\vec{q}\,|))^2 - s)}\;\frac{2\sqrt{s}}{(s - (\omega_1(|\vec{q}\,|)-\omega_2(|\vec{q}\,|)^2)}\;\delta(q^0)$$

The Blankenbecler–Sugar–Logunov–Tavkhelidze–Alessandrini–Omnes (BSLTAO) QP equation (1963, 1965, 1966) is determined by the choice $f(s',s) = 1$. Thomson (1970) suggested $f(s',s) = (\sqrt{s}+\sqrt{s'})/\sqrt{4\,s}$, while Hummel and Tjon (HT)(1990) used $f(s',s) = \sqrt{4\,s'}/(\sqrt{s}+\sqrt{s'})$. The BSE in the "equal-time" (ET) approximation leads after inclusion of crossed–box exchange to the kernel to a QP equation with $f(s',s) = 2-(s/s')$.

(b) Determine $S^{\,\mathrm{QP}}_{\mathrm{scalar}}(P\,,q)$ for the BSLTAO-, Thomson-, HT- and ET-case in the cm-system for $m_1 = m_2 = m$. Which of these QP equations has the correct OPL?

Choosing a prescription going back to F. Gross (1969) one can construct also asymmetrically QP equations by setting only one particle on its mass shell. For the case $m_1 = m_2 = m$ one could define:

$$S^{\,\mathrm{QP}'}_{\mathrm{scalar}}(P,q) = 2\pi\,i\int_{(2\,m)^2}^{\infty} ds'\;\frac{f(s',s)}{s' - s - i\,\varepsilon}\;\delta^{(+)}\big[\,p_1^2 - m^2\,\big]\;\delta^{(+)}\Big[\Big(\sqrt{\frac{s'}{s}}\,P - p_1\Big)^2 - m^2\Big]$$

(c) Evaluate the s'-integration in the cm-frame. What will change in the resulting δ-distribution compared to $\delta(q^0)$ in the standard QP approach?

(d) What will be $S^{\,\mathrm{QP}'}_{\mathrm{scalar}}(P_{cm},q)$ for choice of Erkelenz and Holinde (1972) ($f(s,s') = 1$), by F. Gross (1969) ($f(s',s) = (\sqrt{s}+\sqrt{s'})/\sqrt{4\,s}$) and by Kadyshevsky (1968) ($f(s',s) = (\sqrt{s}+\sqrt{s'})/\sqrt{4\,s'}$).

(e) How can this framework be extended to the case $m_1 \neq m_2$? Is there a well defined OPL of the corresponding QP equations?

(f) How may the QP and OP' construction be extended to a two-Fermion problem?

6) Numerical solution of the homogenous "ladder" BSE of the $\varphi^2\,\chi$-model (II):

Reproduce the results of I. Fukui and N. Setô of hep-ph/9509382 (= Prog. Theor. Phys. 95 (1996) 433) and hep-ph/9901364 (= Prog. Theor. Phys. 101 (1999) 1335) in the case $\ell = 0, 1$, using your computer code of problem 3 for the solution of a complex eigenvalue problem. Take the BS-kernels $K_{LL',\ell}(z,z')$ either from the lecture notes or use the original results for the kernels published in N. Setô and I. Fukui, Prog. Theor. Phys. 89 (1993) 205, based on the work N. Setô, Prog. Theor. Phys. 40 (1968) 605.