Dyson-Schwinger Techniques and the Bethe-Salpeter Equation (Introduction)

7) The homogeneous 2-Fermion BSE with a separable (= degenerate) kernel:

As has been demonstrated in the lecture for Bosonic systems the (Fourier-transformed) Bethe–Salpeter amplitudes (BSAs) and their adjoints in the two (anti-)Fermion sector can be introduced in terms of the Jacobi-coordinates of problem 4 e.g. by:

$$\begin{aligned}
\ll 0|\,T\,[\psi^{(2)}(x_2)\;\psi^{(1)}(x_1)\,]\,|P,B>_c &= e^{-iPX}\int\frac{d^4q}{(2\pi)^4}\;e^{-iqx}\;\psi_B^{(2\,1)}(P,q)\\
\ll 0|\,T\,[\bar\psi^{(2)}(x_2)\;\bar\psi^{(1)}(x_1)\,]\,|P,B>_c &= e^{-iPX}\int\frac{d^4q}{(2\pi)^4}\;e^{-iqx}\;\psi_B^{(\bar2\,\bar1)}(P,q)\\
&\ldots\\
\ll P,B|\,T\,[\bar\psi^{(1)}(x_1)\;\bar\psi^{(2)}(x_2)\,]\,|0>_c &= e^{+iPX}\int\frac{d^4q}{(2\pi)^4}\;e^{+iqx}\;\tilde\psi_B^{(2\,1)}(P,q)\\
&\ldots
\end{aligned}$$

The bound (or scattering) state $|P,B>$ with 4-momentum P and quantum numbers B fulfils the normalization condition $\ll P',B'|P,B>=(2\pi)^3\;2\,\omega_B(|\vec P\,|)\,\delta^3(\vec P\,'-\vec P\,)\;\delta_{B'B}$. The BSAs can be rewritten in terms of connected two-point Green's functions multiplied by BS-vertex functions $\Gamma_B^{(i\,j)}(P,q)$ (& their adjoints $\tilde\Gamma_B^{(i\,j)}(P,q)$) ($i\in\{2,\bar2\}$, $j\in\{1,\bar1\}$):

$$\begin{aligned}
\psi_B^{(2\,1)}(P,q) &= \frac{1}{(\,\not p_2-m_2+\Sigma_2)^{(2)}}\;\frac{1}{(\,\not p_1-m_1+\Sigma_1)^{(1)}}\;\Gamma_B^{(2\,1)}(P,q)\\
\psi_B^{(\bar2\,\bar1)}(P,q) &= \Gamma_B^{(\bar2\,\bar1)}(P,q)\;\frac{1}{(-\,\not p_2-m_2+\bar\Sigma_2)^{(2)}}\;\frac{1}{(-\,\not p_1-m_1+\bar\Sigma_1)^{(1)}}\\
&\ldots\\
\tilde\psi_B^{(2\,1)}(P,q) &= \tilde\Gamma_B^{(2\,1)}(P,q)\;\frac{1}{(\,\not p_2-m_2+\Sigma_2)^{(2)}}\;\frac{1}{(\,\not p_1-m_1+\Sigma_1)^{(1)}}\\
&\ldots
\end{aligned}$$

(Omission of Fermion self-energies yields $\Sigma\to i\,\varepsilon$ and $\bar\Sigma=\gamma_0\,\Sigma^+\gamma_0\to -\,i\,\varepsilon$). Corresponding to problem 4 also for the Fermionic case a BSE can be derived, which is solved by both the BSAs and the corresponding connected Green's functions. For the Fermion Fermion (FF) system it is ($S^{-1}_{\text{Dirac}}(P,q):=(\,\not p_1-m_1+\Sigma_1)^{(1)}(\,\not p_2-m_2+\Sigma_2)^{(2)}$):

$$S_{\text{Dirac}}(P,q)\;\psi_B^{(2\,1)}(P,q)=-\int\frac{d^4\bar q}{(2\pi)^4}\;K(P;q,\bar q)\,\psi_B^{(2\,1)}(P,\bar q)\;.$$

(a) Rewrite this BSE as a BSE for the corresponding vertex function $\Gamma_B^{(2\,1)}(P,q)$.

(b) What are the corresponding BSEs for a Fermion anti-Fermion (FA) system?

(c) Express $\ll 0|\,T\,[\psi^{(2)}(x_2')\;\;\psi^{(1)}(x_1')\,\bar\psi^{(2)}(x_2)\;\;\bar\psi^{(1)}(x_1)\,]\,|0>$ close to the pole by $\psi_B^{(2\,1)}(P,q)$ and $\tilde\psi_B^{(2\,1)}(P,q)$. Reformulate the same procedure for the FA-case.

(d) Derive normalization conditions for $\psi_B^{(2\,1)}(P,q)$, $\psi_B^{(2\,\bar1)}(P,q)$, $\Gamma_B^{(2\,1)}(P,q)$ and $\Gamma_B^{(2\,\bar1)}(P,q)$.

(e) Discuss, why the most general *local* parity preserving FF-kernel has the structure $K(P;q,\bar q)=\sum\limits_n K_n\,\Gamma_n$ with $\Gamma_n\in\{1_4^{(1)}\;1_4^{(2)},\gamma_5^{(1)}\;\gamma_5^{(2)},\gamma_\mu^{(1)}\;\gamma^{(2)\,\mu},[\gamma_5\gamma_\mu]^{(1)}\;[\gamma_5\gamma^\mu]^{(2)},\sigma_{\mu\nu}^{(1)}\;\sigma^{(2)\,\mu\nu}\}$ and $K_n=K_n(P^2,q^2,\bar q^2,P\cdot q,P\cdot\bar q,q\cdot\bar q)$ = complex function. How to extend this ansatz to *non-local* interactions? What will change in the FA-case?

(f) Reformulate the BSE for a separable BS-kernel $K(P;q,\bar q)=-\,i\,\sum\limits_{ij}\sum\limits_n V_{i,n}(\bar q\,)\lambda^n_{ij}\,V_{j,n}(q)\,\Gamma_n$ as an eigenvalue problem (FF&FA). Determine the BSA by the eigensolution.

[1] CFIF/IST, Edifício Ciência, Piso 4, Gab. 4-8.7., Tel. 218419107 or 3107, kleefeld@cfif.ist.utl.pt

[2] 1st course in "QCD and more", CFIF postgraduate lecture courses 2001

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8) The homogeneous 2-Boson BSE with a separable (= degenerate) kernel:

Starting point is the *homogenous* 2-Boson BSE (problem 4 (b)) for the BSA or BS-vertex function of a 2-Boson system, which can be introduced like in problem 7 by:

$$\begin{aligned}
\ll 0|\,T\,[\,\varphi_i(x_2)\;\varphi_j(x_1)\,]\,|P,B>_c &= e^{-iPX}\int\frac{d^4q}{(2\pi)^4}\;e^{-iqx}\;\psi_{B,\,ij}(P,q)\\
\ll P,B|\,T\,[\,\varphi_j(x_1)\;\varphi_i(x_2)\,]\,|0>_c &= e^{+iPX}\int\frac{d^4q}{(2\pi)^4}\;e^{+iqx}\;\tilde{\psi}_{B,\,ij}(P,q)\\
\psi_{B,\,ij}(P,q) &= \frac{1}{(p_2^2-\,m_2^2+\,\Pi_2)}\,\frac{1}{(p_1^2-\,m_1^2+\,\Pi_1)}\;\Gamma_{B,\,ij}(P,q)\\
\tilde{\psi}_{B,\,ij}(P,q) &= \tilde{\Gamma}_{B,\,ij}(P,q)\;\frac{1}{(p_2^2-\,m_2^2+\,\Pi_2)}\,\frac{1}{(p_1^2-\,m_1^2+\,\Pi_1)}
\end{aligned}$$

Like in problem 7 the generalization to Boson anti-Boson and anti-Boson anti-Boson systems is straight forward. Omission of Bosonic self-energies: $\Pi\to i\,\varepsilon$, $\bar{\Pi}=\Pi^*\to -\,i\,\varepsilon$.

(b) Determine the coupling λ of the *homogenous* 2-Boson BSE (problem 4) for a separable BS-kernel $K(P;q,\bar{q}\,)=-\,i\,\lambda\,V(q)\,V(\bar{q})$ in terms of $S_{\text{scalar}}(P,q)$, $V(q)$, $V(\bar{q})$.

(c) Reformulate the BSE for a separable BS-kernel $K(P;q,\bar{q})=-\,i\,\sum\limits_{ij}V_i(\bar{q}\,)\lambda_{ij}\,V_j(q)$ (λ_{ij} invertible) as an eigenvalue problem. Determine the BSA by the eigensolution.

9) Generalized spinors and their properties:

For the description of 2-(anti)Fermion systems it is convenient to compactify the notation of Dirac-spinors. In my convention this is done by the following definitions:

$$u_{+1}(\vec{p},s)\;:=\;u\,(\vec{p},s)\;,\quad u_{-\,1}(\vec{p},s)\;:=\;v\,(\vec{p},s)\;\Rightarrow\;u_{\,\alpha}(\vec{p},s)\;\stackrel{!}{=}\;\gamma_5\,u_{-\,\alpha}(\vec{p},-s)\qquad(\alpha\,=\,\pm1)$$

$$U_\alpha(\vec{p},s)\;:=\;\frac{u_{\,\alpha}(\alpha\,\vec{p},s)}{\sqrt{2\,\omega(|\vec{p}\,|)}}\quad(\alpha\,=\,\pm1)\quad\Rightarrow\quad u_{\,\alpha}(\vec{p},s)\;=\;\sqrt{2\,\omega(|\vec{p}\,|)}\;U_{\,\alpha}(\alpha\,\vec{p},s)$$

The standard projectors $\Lambda_{\,\pm1}(\vec{p}\,)$ on positive/negative energy states are given by:

$$\Lambda_{\,\alpha}(\vec{p}\,)\;:=\;(\omega\,(|\vec{p}\,|)+\alpha\,H(\vec{p}\,))/(2\,\omega\,(|\vec{p}\,|))\quad(\alpha\,=\,\pm\,1)\quad\text{with}\quad H(\vec{p}\,)\;=\;\vec{\alpha}\cdot\vec{p}+\beta\,m$$

Show that the following identities hold:

(a) $\sum\limits_s u_{\,\alpha}\,(\vec{p},s)\;\bar{u}_{\,\alpha}\,(\vec{p},s)\;=\;(\,\not{p}+\alpha\,m)|_{p^0=\omega\,(|\vec{p}\,|)}\;\stackrel{!}{=}\;2\,\omega\,(|\vec{p}\,|)\,\Lambda_{\,\alpha}(\alpha\,\vec{p}\,)\,\gamma_0\quad(\alpha\,=\,\pm1)$

(b) $2\,\omega\,(|\vec{p}\,|)\;\Lambda_{\,\alpha}(\vec{p}\,)\,\gamma_0=\sum\limits_s u_{\,\alpha}\,(\alpha\,\vec{p},s)\;\bar{u}_{\,\alpha}\,(\alpha\,\vec{p},s)=\begin{cases}(\,\not{p}+m)|_{p^0=\omega\,(|\vec{p}\,|)} & (\alpha=+1)\\ \gamma_0\,(\,\not{p}-m)\,\gamma_0\,|_{p^0=\omega\,(|\vec{p}\,|)} & (\alpha=-1)\end{cases}$

(c) $U^+_{\alpha'}(\vec{p},s')\,U_\alpha(\vec{p},s)\;=\;\delta_{\alpha'\alpha}\;\delta_{s's}\;,\quad\sum\limits_{\alpha,s}U_\alpha(\vec{p},s)\,U^+_\alpha(\vec{p},s)\;=\;1_4$

(d) $\dfrac{\not{p}+m}{p^2-m^2+i\,\varepsilon}\;\stackrel{!}{=}\;\sum\limits_\alpha\dfrac{\alpha\,\Lambda_{\,\alpha}(\vec{p}\,)\,\gamma_0}{\alpha\,p^0-\omega(|\vec{p}\,|)+i\,\varepsilon}\;\stackrel{!}{=}\;\dfrac{1}{2\,\omega\,(|\vec{p}\,|)}\sum\limits_{\alpha,s}\dfrac{\alpha\;u_{\,\alpha}\,(\alpha\,\vec{p},s)\;\bar{u}_{\,\alpha}\,(\alpha\,\vec{p},s)}{\alpha\,p^0-\omega(|\vec{p}\,|)+i\,\varepsilon}$

(e) $\Lambda_{\,\alpha}(\vec{p}\,)\;\stackrel{!}{=}\;\sum\limits_s U_{\,\alpha}\,(\vec{p},s)\;U^+_{\alpha}\,(\vec{p},s)$

(f) $\not{p}+m\;\stackrel{!}{=}\;\sum\limits_\alpha\,(p^0+\alpha\,\omega(|\vec{p}\,|))\;\Lambda_{\,\alpha}(\vec{p}\,)\;\gamma_0\stackrel{!}{=}\sum\limits_{\alpha,s}\,(p^0+\alpha\,\omega(|\vec{p}\,|))\;U_{\,\alpha}\,(\vec{p},s)\;\bar{U}_{\,\alpha}\,(\vec{p},s)$

(g) $\dfrac{\not{p}+m}{p^2-m^2+i\,\varepsilon}\;\stackrel{!}{=}\;\sum\limits_{\alpha,s}\dfrac{\alpha\,U_{\,\alpha}\,(\vec{p},s)\;\bar{U}_{\,\alpha}\,(\vec{p},s)}{\alpha\,p^0-\omega(|\vec{p}\,|)+i\,\varepsilon}$