Dyson-Schwinger Techniques and the Bethe-Salpeter Equation (Introduction)

10) Wigner rotation and spin projectors in the Fermion-Fermion system:

From Quantum Mechanics (QM) it is well known, that a right–handed rotation of a configuration space 3-vector $\vec{x}$ by an (positive) angle $|\vec{\varphi}\,|$ with respect to a rotation axis $\vec{\varphi}$ into a vector $\vec{x}\,'$ is performed by:

$$\vec{x}\,' \quad = \quad \exp\,(-\,\frac{i}{\hbar}\,\vec{\varphi}\cdot\vec{L}\,)\;\vec{x}$$

while $\vec{L}$ is the orbital angular momentum operator. The corresponding rotation operator for the state vector of one spin 1/2 particle is of course given by:

$$\exp\,(-\,\frac{i}{2}\,\vec{\varphi}\cdot\vec{\sigma}\,) \quad \stackrel{!}{=} \quad 1_2\,\cos\frac{|\vec{\varphi}\,|}{2} \;-\; i\;\frac{\vec{\varphi}\cdot\vec{\sigma}}{|\vec{\varphi}\,|}\;\sin\frac{|\vec{\varphi}\,|}{2}$$

A Wigner rotation of a quantum mechanical two Fermion system is of course performed by $R\,(\vec{x}\,',\vec{x}\,) := \exp\,(-\,\frac{i}{2}\,\vec{\varphi}\cdot\vec{\sigma}^{\,(1)}\,)\;\exp\,(-\,\frac{i}{2}\,\vec{\varphi}\cdot\vec{\sigma}^{\,(2)}\,)$.

(a) Show, that the generalisation of this operator acting on two Dirac spinors can be expressed by:

$$\begin{aligned}
\mathcal{R}\,(\vec{x}\,',\vec{x}\,) \quad &= \quad \frac{1}{2\,|\vec{x}\,|^2\,|\vec{x}\,'\,|^2}\;\frac{|\vec{x}\,|\,|\vec{x}\,'\,| + \vec{x}\cdot\vec{x}\,'}{|\vec{x}\,|\,|\vec{x}\,'\,|} \\
&\cdot\;\left(1_4\,|\vec{x}\,|\,|\vec{x}\,'\,| \;-\; i\;\frac{|\vec{x}\,|\,|\vec{x}\,'\,|}{|\vec{x}\,|\,|\vec{x}\,'\,| + \vec{x}\cdot\vec{x}\,'}\;\;[\vec{x}\times\vec{x}\,'\,]\cdot[\vec{\gamma}\,\gamma_5]\right)^{(1)} \\
&\cdot\;\left(1_4\,|\vec{x}\,|\,|\vec{x}\,'\,| \;-\; i\;\frac{|\vec{x}\,|\,|\vec{x}\,'\,|}{|\vec{x}\,|\,|\vec{x}\,'\,| + \vec{x}\cdot\vec{x}\,'}\;\;[\vec{x}\times\vec{x}\,'\,]\cdot[\vec{\gamma}\,\gamma_5]\right)^{(2)}
\end{aligned}$$

In a nucleon-nucleon system (e.g. like the deuteron) we can define the corresponding projectors on spin one states $\mathcal{P}_1$ and spin zero states $\mathcal{P}_0$ by:

$$\begin{aligned}
\mathcal{P}_1 \;&:=\; \frac{3}{4}\,1_4^{(1)}\,1_4^{(2)} \;+\; \frac{1}{4}\,[\vec{\gamma}\,\gamma_5\,]^{(1)}\cdot[\vec{\gamma}\,\gamma_5\,]^{(2)} \\
\mathcal{P}_0 \;&:=\; \frac{1}{4}\,1_4^{(1)}\,1_4^{(2)} \;-\; \frac{1}{4}\,[\vec{\gamma}\,\gamma_5\,]^{(1)}\cdot[\vec{\gamma}\,\gamma_5\,]^{(2)}
\end{aligned}$$

(b) Show, that the following commutation and multiplication identities hold:

$$\begin{aligned}
0 \;&=\; \frac{1}{2}\,(\,1_4+\gamma_0\,)^{(1)}\;\frac{1}{2}\,(\,1_4+\gamma_0\,)^{(2)}\;[\,\mathcal{P}_1\;\;\mathcal{R}\,(\vec{x}\,',\vec{x}\,)\;-\;\mathcal{R}\,(\vec{x}\,',\vec{x}\,)\;\;\mathcal{P}_1\,] \\
0 \;&=\; \frac{1}{2}\,(\,1_4+\gamma_0\,)^{(1)}\;\frac{1}{2}\,(\,1_4+\gamma_0\,)^{(2)}\;\Big[\mathcal{S}_{12}(\hat{\vec{x}}\,'\,)\;\;\mathcal{R}\,(\vec{x}\,',\vec{x}\,)\;-\;\mathcal{R}\,(\vec{x}\,',\vec{x}\,)\;\;\mathcal{S}_{12}(\hat{\vec{x}}\,)\Big] \\
0 \;&=\; \frac{1}{2}\,(\,1_4+\gamma_0\,)^{(1)}\;\frac{1}{2}\,(\,1_4+\gamma_0\,)^{(2)}\;[\,\mathcal{P}_1\,\mathcal{P}_1\;-\;\mathcal{P}_1\,] \\
0 \;&=\; \frac{1}{2}\,(\,1_4+\gamma_0\,)^{(1)}\;\frac{1}{2}\,(\,1_4+\gamma_0\,)^{(2)}\;\Big[\mathcal{S}_{12}(\hat{\vec{x}}\,)\;\mathcal{S}_{12}(\hat{\vec{x}}\,) + 2\,\mathcal{S}_{12}(\hat{\vec{x}}\,) \;-\; 8\,\mathcal{P}_1\Big] \\
0 \;&=\; \frac{1}{2}\,(\,1_4+\gamma_0\,)^{(1)}\;\frac{1}{2}\,(\,1_4+\gamma_0\,)^{(2)}\;\Big[\mathcal{S}_{12}(\hat{\vec{x}}\,)\;\mathcal{P}_1\;-\;\mathcal{S}_{12}(\hat{\vec{x}}\,)\Big] \\
0 \;&=\; \frac{1}{2}\,(\,1_4+\gamma_0\,)^{(1)}\;\frac{1}{2}\,(\,1_4+\gamma_0\,)^{(2)}\;\Big[\mathcal{P}_1\;\mathcal{S}_{12}(\hat{\vec{x}}\,)\;-\;\mathcal{S}_{12}(\hat{\vec{x}}\,)\Big]
\end{aligned}$$

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[2] 1st course in “QCD and more”, CFIF postgraduate lecture courses 2001

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11) Proper vertex functions, WT identities, linear/quadratic symmetry breaking:

As discussed in the lecture, the effective action $S\,[<\vec{\varphi}>_c] = S^{(0)}\,[<\vec{\varphi}>_c] + O(\hbar)$ is connected to the generating functional of the connected Green's functions $W[\vec{j}\,]$ by a Legendre transformation, i.e.:

$$W[\vec{j}\,] = S\,[<\vec{\varphi}>_c] + \int d^4x\, \vec{j}(x)\cdot \underbrace{<\vec{\varphi}(x\,;\vec{j}\,)>_c}_{\vec{G}_{1\,c}(x;\vec{j}\,)}\;;\; <\vec{\varphi}(x\,;\vec{j}\,)>_c = \frac{\delta W[\vec{j}\,]}{\delta \vec{j}(x)}\;;\; \vec{j}(x) = -\,\frac{\delta S\,[<\vec{\varphi}>_c]}{\delta <\vec{\varphi}(x\,;\vec{j}\,)>_c}\,.$$

The effective action S is the generating functional for the "(proper) vertex functions" $\Gamma_{n\,c\,;\,i_1\ldots i_n}(x_1,\ldots,x_n\,) := \Gamma_{n\,c\,;\,i_1\ldots i_n}(x_1,\ldots,x_n;\vec{j}\,)|_{\vec{j}=\vec{0}}$, i.e.:

$$S\,[<\vec{\varphi}>_c] =$$
$$= \sum_n \frac{1}{n!} \sum_{i_1,\ldots,i_n} \int d^4x_1 \ldots d^4x_n\; \Gamma_{n\,c\,;\,i_1\ldots i_n}(x_1,\ldots,x_n;\vec{j}\,)\; <\varphi_{i_1}(x_1;\vec{j}\,)>_c \ldots <\varphi_{i_n}(x_n;\vec{j}\,)>_c$$

A symmetry of the effective action under an infinitesimal transformation of the fields, i.e. $\vec{\varphi}(x) = \vec{\chi}(x) + \varepsilon\,\vec{F}(x;\vec{\chi}\,)$, yields non-trivial constraints on its Green's and vertex functions, which are usually called Ward-Takahashi (WT) identities (in the case of anti-commuting or non-linear type symmetries like e.g. BRS- or SUSY-transformations Slavnov-Taylor identities), yielding in the classical limit the conservation of a Noether-current, whose charges act as the generators of the continuous symmetry group (see e.g. J. Zinn-Justin (©1996), "QFT and Critical Phenomena", Oxford Univ. Press). In the above case the generators are the Taylor-coefficients $\vec{F}_{n\,;\,i_1\ldots i_n}(x;x_1,\ldots,x_n)$ of $\vec{F}(x;\vec{\chi}\,)$:

$$\vec{F}(x;\vec{\chi}\,) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1,\ldots,i_n} \int d^4x_1 \ldots d^4x_n\; \vec{F}_{n\,;\,i_1\ldots i_n}(x;x_1,\ldots,x_n)\; \chi_{i_1}(x_1)\ldots\chi_{i_n}(x_n)$$

In case of symmetry the change of Bosonic variables leaves the structure of the partition function upto $O(\varepsilon^2)$ invariant, i.e. $\int D\vec{\chi}\,\det\left[\frac{\delta\vec{\varphi}}{\delta\vec{\chi}}\right]\,\exp\left[\frac{i}{\hbar}\left(S^{(0)}[\vec{\varphi}] + \int d^4x\,\vec{\varphi}(x)\cdot\vec{j}(x)\right)\right] = \int D\vec{\chi}\,\exp\left[\frac{i}{\hbar}\left(S^{(0)}[\vec{\chi}] + \int d^4x\,\vec{\chi}(x)\cdot\vec{j}(x)\right)\right] + O(\varepsilon^2)$.

(a) Show that upto $O(\varepsilon)$ this identity yields the same WT-identity, which one would obtain, if one applies $\int d^4x\,\vec{F}(x\,;\frac{\hbar}{i}\frac{\delta}{\delta\vec{j}})$ to Schwinger's equation $\frac{\delta\,T[\vec{j}]}{\delta\,\varphi(y)} = 0$. For the ε-expansion of $\det\frac{\delta\vec{\varphi}}{\delta\vec{\chi}}$ use $\ln\det\frac{\delta\vec{\varphi}}{\delta\vec{\chi}} = \mathrm{tr}\ln\frac{\delta\vec{\varphi}}{\delta\vec{\chi}}$, $\mathrm{tr}\ln[1+M] = \sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n}\,\mathrm{tr}[M^n]$, $\mathrm{tr}[\,M^n\,] := \sum\limits_{i_1,\ldots,i_n}\int d^4x_1\ldots d^4x_n\; M_{i_1\,i_2}(x_1,x_2)\; M_{i_2\,i_3}(x_2,x_3)\;\ldots\; M_{i_n\,i_1}(x_n,x_1)$ and $M_{k\ell}(x,y) := \frac{\delta\varphi_k(x)}{\delta\chi_\ell(y)} - \delta_{k\ell}\,\delta^4(x-y)$.

(b) Derive for the symmetry transformation $\varphi_i(x) = \chi_i(x) + \sum\limits_{j,\alpha}\varepsilon_\alpha\,t^{\,\alpha}_{ij}\,\chi_j(x)$ the corresponding WT-identities for the one-point Green's and vertex functions. How to extend these WT-identities in momentum space to n-point vertex functions?

(c) Linear symmetry breaking is introduced by replacing the symmetric action $S^{(0)}_{sym}$ by $S^{(0)}_{sym} - \int d^4x\,\vec{A}\cdot\vec{\varphi}\,(x)$. What will change in the results of problem 11 (b)?

(d) Quadratic symmetry breaking is introduced by replacing the symmetric action $S^{(0)}_{sym}$ by $S^{(0)}_{sym} + \frac{1}{2}\sum\limits_{ij}\int d^4x\,\mu_{\,ij}\,\varphi_i\,(x)\,\varphi_j\,(x)$. What will change in 11 (b)?